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APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATION OF A STEP FUNCTION SP--ETC(U)

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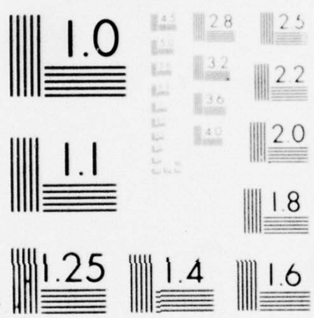

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APPROXIMATE MAXIMUM LIKELIHOOD
ESTIMATION OF A STEP FUNCTION
SPECTRAL DENSITY

by

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SUMMARY

Approximate maximum likelihood estimates are obtained for the ordinates of a step function spectral density in the Gaussian case. The estimates are simply integral averages of the periodogram over the frequency bands in which the density is constant. Whittle's form of the approximate likelihood is used. Results are given for scalar and vector processes.

Some key words: Stationary Gaussian process; Step function spectral density; Periodogram average; Approximate maximum likelihood estimates.

1. Introduction

Estimates of the spectral density of a stationary process are typically constructed by smoothing the periodgram. The motivation for this procedure is well-known. For a large sample size and under suitable conditions the expectation of the periodogram is approximately equal to the spectral density. However, the variance does not tend to zero as the sample size increases. Averaging of the periodogram reduces the variance and is a method which can be used to produce a consistent estimate. At the same time the averaging may increase the bias.

Spectral estimates constructed by periodogram averaging are essentially estimates obtained by the method of moments. They are formed by replacing population covariances by their sample analogues and then smoothing for the purpose of stabilizing the variance. In this paper we shall assume a Gaussian model and show that periodogram averages are approximate maximum likelihood estimates of certain spectral parameters. The spectral density is assumed to be a positive step function with a finite number of distinct ordinates. These ordinates are the spectral parameters to be estimated.

To derive the results we shall use the approximate Gaussian likelihood proposed by Whittle (1953 a,b, 1954) for the analysis of finite parameter linear time series models. The use of a step function spectral density in Whittle's approximate likelihood leads to a simplification which permits easy exact solution of the approximate maximum likelihood derivative equations. The resulting

estimates are in fact estimates of the average amount of power in each of a set of bands determined by a finite partition of the frequency axis.

The approximate likelihood to be used is discussed in §2 and results for scalar time series are derived in §3. The vector case is treated in §4.

2. An Approximate Likelihood for a Gaussian Time Series

Let $\{x_t,\ t=0, \pm 1, \ldots\}$ be a stationary Gaussian process with mean 0 and spectral density $f(\lambda)(-\pi \leq \lambda \leq \pi)$. We assume throughout that $f(\lambda)$ is positive. The problem to be considered is estimation of $f(\lambda)$ or of parameters associated with $f(\lambda)$ when an observation of $\underset{\sim}{x} = (x_1, \ldots, x_T)'$ is available. The exact density of $\underset{\sim}{x}$ is

$$(2\pi)^{-\frac{1}{2}T} |\underset{\sim}{\Sigma}|^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\underset{\sim}{x}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{x}), \tag{1}$$

where $\underset{\sim}{\Sigma}$ is the covariance matrix of $\underset{\sim}{x}$.

Let $\{\epsilon_t,\ t=0, \pm 1, \ldots\}$ be a sequence of independent variables, each with mean 0 and variance σ^2. Then the process $\{x_t\}$ defined by

$$x_t = \sum_{j=0}^{\infty} \delta_j \epsilon_{t-j} \qquad (t=0, \pm 1, \ldots), \tag{2}$$

where $\delta_0 = 1$ and $\Sigma_{j=0}^{\infty} \delta_j^2 < \infty$, is termed a linear process. The spectral density associated with (2) is

$$f(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} \delta_j e^{i\lambda j} \right|^2 \qquad (-\pi \leq \lambda \leq \pi).$$

It is common to assume that each δ_j in (2) is a function of a vector of parameters about which inference is desired. This is the case, for example, when $\{x_t\}$ is an ARMA (p,q) process,

$$\sum_{j=0}^{p} \alpha_j x_{t-j} = \sum_{k=0}^{q} \beta_k \epsilon_{t-k} \quad (t=0, \pm 1, \ldots),$$

where $\alpha_0 = \beta_0 = 1$.

For the case of Gaussian models of the form (2) Whittle (1953a,1954) replaced (1) by

$$(2\pi\sigma^2)^{-\frac{1}{2}T} \exp\left\{- \frac{T}{4\pi} \int_{-\pi}^{\pi} \frac{I(\lambda)}{f(\lambda)} d\lambda\right\}$$

$$= (2\pi)^{-T} \exp\left[- \frac{T}{4\pi} \int_{-\pi}^{\pi} \left\{ \log f(\lambda) + \frac{I(\lambda)}{f(\lambda)}\right\} d\lambda\right] , \tag{3}$$

where

$$I(\lambda) = \frac{1}{2\pi T} \left| \sum_{t=1}^{T} x_t e^{i\lambda t} \right|^2 \quad (-\pi \leq \lambda \leq \pi)$$

is the periodogram.

Two considerations are involved in the approximation (3). One is the replacement of $\underset{\sim}{\Sigma}^{-1}$ by the matrix $\underset{\sim}{\Sigma}^*$ defined by

$$\underset{\sim}{\Sigma}^* = \left(\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} e^{i\lambda(s-t)} f^{-1}(\lambda) d\lambda\right)_{s,t=1,\ldots,T} ,$$

which leads directly to the exponent in the first line of (3). It can be shown that in fact $\underset{\sim}{\Sigma}^* - \underset{\sim}{\Sigma}^{-1}$ is positive semidefinite for every stationary process of the form (2) with a positive spectral density. Moreover, the rank of $\underset{\sim}{\Sigma}^* - \underset{\sim}{\Sigma}^{-1}$ is $\min\{2\max(p,q),T\}$ for an ARMA (p,q) process and is T if $\{x_t\}$ of the form (2) is not an ARMA (p,q) process with p and q both finite. (See Shaman, 1976.)

The other feature of the approximation (3) is the replacement of $|\underset{\sim}{\Sigma}|^{\frac{1}{2}}$ by $(\sigma^2)^{\frac{1}{2}T}$. Note that the second line of (3) follows from the first by

$$\sigma^2 = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\{2\pi f(\lambda)\}d\lambda\right] ,$$

where σ^2 is the variance of the one-step-ahead prediction error and is the same as the variance of ϵ_t in (2). (See, e.g., Grenander and Rosenblatt, 1957, p. 69.)

The replacement of $|\underset{\sim}{\Sigma}|^{\frac{1}{2}}$ by $(\sigma^2)^{\frac{1}{2}T}$ may be motivated as follows. It is well-known (Grenander and Rosenblatt, 1957, pp. 103-5) that the eigenvalues of the Toeplitz matrix $\underset{\sim}{\Sigma}$ are approximately given by equally-spaced spectral ordinates. Then, denoting the eigenvalues by λ_j $(j=1,\ldots,T)$, we can write

$$\begin{aligned}\log|\underset{\sim}{\Sigma}| &= \sum_{j=1}^{T} \log\lambda_j \\ &\approx \sum_{j=1}^{T} \log\left\{2\pi f\left(\frac{2\pi j}{T}\right)\right\} ,\end{aligned}$$

and the latter sum is approximately

$$\frac{T}{2\pi}\int_{-\pi}^{\pi} \log\{2\pi f(\lambda)\}d\lambda .$$

This heuristic argument for the approximation of $|\underset{\sim}{\Sigma}|^{\frac{1}{2}}$ is not dependent upon the representation of $\{x_t\}$ in (2), but holds for any process for which $\int_{-\pi}^{\pi} \log f(\lambda)d\lambda > -\infty$. Moreover, the same argument indicates that the eigenvalues of $\underset{\sim}{\Sigma}^{-1}$ are approximately given by equally-spaced values of $f^{-1}(\lambda)$. The matrix $\underset{\sim}{\Sigma}^*$ may be taken as an approximation to $\underset{\sim}{\Sigma}^{-1}$ for processes $\{x_t\}$ more general than (2). In fact, $\underset{\sim}{\Sigma}^{-1} = \underset{\sim}{\Sigma}^*$ if both are (two-sided) infinite-dimensional matrices. That is,

$$\sum_{r=-\infty}^{\infty} \sigma(s-r)\sigma^*(r-t) = \delta(s-t) \quad (s,t=0,\pm 1,\ldots), \tag{4}$$

where $\delta(\cdot)$ is the Kronecker delta and $\sigma(\cdot)$ and $\sigma^*(\cdot)$ are the elements of the Toeplitz matrices $\underset{\sim}{\Sigma}$ and $\underset{\sim}{\Sigma}^*$, respectively. The result (4) follows because the left side of (4) is the convolution of two covariance sequences, and the Fourier transform of this convolution is 2π times the product of the corresponding spectral densities.

For the above reasons we shall use the approximation to (1) given by the second line of (3) when $\{x_t\}$ is any stationary process with a positive spectral density. The vector case analogue of this approximation will be used in §4.

3. Periodogram Averages as Approximate Maximum Likelihood Estimates.

In this section we assume the spectral density is a step function and derive approximate maximum likelihood estimates for

the ordinates of the density. Let

$$f(\lambda) = \sum_{j=1}^{q} f_j c_j(\lambda) \quad (0 \le \lambda \le \pi), \tag{5}$$

where

$$c_j(\lambda) = 1 \quad (\lambda_{j-1} \le \lambda < \lambda_j)$$

$$= 0 \quad \text{(otherwise)},$$

$f(-\lambda) = f(\lambda)$, and $0 = \lambda_0 < \lambda_1 < \dots < \lambda_q = \pi$. Set $f(\pi) = f_q$.
Since

$$f^{-1}(\lambda) = \sum_{j=1}^{q} \frac{1}{f_j} c_j(\lambda)$$

and $I(\lambda)$ is symmetric, the approximate density (3) becomes

$$(2\pi)^{-T} \exp\left[-\frac{T}{2\pi}\sum_{j=1}^{q}\left\{\log f_j \int_0^{\pi} c_j(\lambda)d\lambda + \frac{1}{f_j}\int_0^{\pi} I(\lambda)c_j(\lambda)d\lambda\right\}\right]. \tag{6}$$

Let L denote the corresponding approximate log likelihood. Then

$$\frac{\partial L}{\partial f_j} = -\frac{T}{2\pi}\int_0^{\pi}\left\{\frac{c_j(\lambda)}{f_j} - \frac{I(\lambda)c_j(\lambda)}{f_j^2}\right\} d\lambda \quad (j=1,\dots,q),$$

and approximate maximum likelihood estimates are given by

$$\hat{f}_j = \frac{\int_0^{\pi} I(\lambda)c_j(\lambda)d\lambda}{\int_0^{\pi} c_j(\lambda)d\lambda} = \frac{1}{\lambda_j - \lambda_{j-1}}\int_{\lambda_{j-1}}^{\lambda_j} I(\lambda)d\lambda \tag{7}$$

$$(j=1,\dots,q).$$

Now we consider asymptotic properties of the estimates (7). Care is required in the analysis because $f(\lambda)$ is not continuous. First we note that $\mathcal{E}\hat{f}_j$ converges to f_j as $T \to \infty$. To see this write

$$\mathcal{E}\int_{\lambda_{j-1}}^{\lambda_j} I(\lambda)\,d\lambda - (\lambda_j - \lambda_{j-1}) f_j$$

$$= \int_{\lambda_{j-1}}^{\lambda_j}\int_{-\pi}^{\pi} F_T(\lambda-\nu) f(\nu)\,d\nu\,d\lambda - \int_{\lambda_{j-1}}^{\lambda_j} f(\lambda)\,d\lambda \qquad (8)$$

$$= \int_{-\pi}^{\pi} F_T(\nu) \int_{\lambda_{j-1}}^{\lambda_j} \{f(\nu+\lambda) - f(\lambda)\}\,d\lambda\,d\nu,$$

where

$$F_T(\nu) = \frac{\sin^2 \frac{1}{2}T\nu}{2\pi T \sin^2 \frac{1}{2}\nu}$$

is the Fejér kernel. The convergence follows as in Theorem 1.5.1 of Dym and McKean (1972). Although $f(\lambda)$ is not continuous, the inner integral in the last line of (8) is a continuous function of ν. From this fact and properties of the Fejér kernel, we obtain the stronger result that $T^{\alpha}(\hat{f}_j - f_j)$ converges to 0 for $0 < \alpha < 1$.

Next consider

$$T\,\mathcal{E}(\hat{f}_j - f_j)^2 = T\,\mathrm{Var}\,\hat{f}_j + T(\mathcal{E}\hat{f}_j - f_j)^2.$$

The second term on the right-hand side tends to 0 as $T \to \infty$, and

$$T(\lambda_j - \lambda_{j-1})^2 \operatorname{Var} \hat{f}_j = T \int_{\lambda_{j-1}}^{\lambda_j} \int_{\lambda_{j-1}}^{\lambda_j} \operatorname{Cov}\{I(\lambda), I(\nu)\} d\lambda d\nu$$

$$= T \int_{\lambda_{j-1}}^{\lambda_j} \int_{\lambda_{j-1}}^{\lambda_j} \left[\left\{ \int_{-\pi}^{\pi} F_T(\mu+\lambda, \mu-\nu) f(\mu) d\mu \right\}^2 \right. \tag{9}$$

$$\left. + \left\{ \int_{-\pi}^{\pi} F_T(\mu+\lambda, \mu+\nu) f(\mu) d\mu \right\}^2 \right] d\lambda d\nu,$$

where

$$F_T(\lambda, \nu) = \frac{\sin \frac{1}{2}\lambda T \sin \frac{1}{2}\nu T}{2\pi T \sin \frac{1}{2}\lambda \sin \frac{1}{2}\nu}$$

(see Anderson, 1971, Theorem 8.2.8). The second summand in the last form of (9) is

$$T \int_{\lambda_{j-1}}^{\lambda_j} \int_{\lambda_{j-1}-\lambda}^{\lambda_j-\lambda} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F_T(x, x+\theta) F_T(y, y+\theta) f(x-\lambda) f(y-\lambda) dx dy d\theta d\lambda.$$

A detailed evaluation of the limit of this expression is lengthy and tedious. One follows the same steps as in, e.g., Theorem 9.3.1 or Problem 16 of Chapter 9 in Anderson (1971), except for an adjustment to account for the fact that $f(\lambda)$ is not assumed to be continuous here. This adjustment consists of observing that

$$\int_{\lambda_{j-1}}^{\lambda_j} f(x-\lambda) f(y-\lambda) d\lambda$$

is uniformly continuous in x and y. The limit of the first summand in the last form of (9) is 0. It follows that

$$\lim_{T\to\infty} T(\lambda_j - \lambda_{j-1})^2 \operatorname{Var} \hat{f}_j = 2\pi \int_{\lambda_{j-1}}^{\lambda_j} f^2(\lambda)\,d\lambda,$$

or

$$\lim_{T\to\infty} T \operatorname{Var} \hat{f}_j = \frac{2\pi f_j^2}{\lambda_j - \lambda_{j-1}} \qquad (j=1,\ldots,q).$$

Moreover,

$$\lim_{T\to\infty} T \operatorname{Cov} (\hat{f}_j, \hat{f}_k) = 0 \qquad (j\neq k).$$

We make several comments. First, the main purpose of the calculations in this section has been to demonstrate that periodogram averages emerge as approximate maximum likelihood estimates of the ordinates of a step function spectral density in the Gaussian case. If the density is in fact a step function, it is unlikely, though, that the statistician desiring to estimate the ordinates will know the correct partition $(\lambda_j, j=0,1,\ldots,q)$. For frequencies ν and μ with $\nu > \mu$ the average

$$\frac{1}{\nu-\mu} \int_{\mu}^{\nu} I(\lambda)\,d\lambda$$

has expected value which converges to the same expression with $f(\lambda)$ in place of $I(\lambda)$, and T times the variance converges to

$$\frac{2\pi}{(\nu-\mu)^2} \int_{\mu}^{\nu} f^2(\lambda)\,d\lambda.$$

However, spectral densities encountered in practice are best viewed as being continuous. Thus, the model (5) really should be regarded as an approximation to a continuous density. One may

interpret this approximation as having arisen from smearing of the original continuous density. The model (5) will be difficult to employ in practice unless a reasonable partition for $[0,\pi]$ is formulated a priori.

A final comment is that the asymptotic information matrix corresponding to the approximate likelihood (6) is diagonal with elements

$$-\lim_{T\to\infty} \mathcal{E}\frac{1}{T}\frac{\partial^2 L}{\partial f_j^2} = -\lim_{T\to\infty} \mathcal{E}\frac{1}{2\pi}\int_0^{\pi}\left\{\frac{c_j(\lambda)}{f_j^2} - \frac{2I(\lambda)c_j(\lambda)}{f_j^3}\right\}d\lambda$$

$$= \frac{\lambda_j - \lambda_{j-1}}{2\pi f_j^2} \qquad (j=1,\ldots,q).$$

Thus the estimates $\hat{f}_j$ are asymptotically efficient in this sense.

4. The Vector Case

The results above extend to the vector case. Let $\{\underset{\sim}{x}_t\}$ be an m-dimensional Gaussian process with mean $\underset{\sim}{0}$ and spectral density $\underset{\sim}{f}(\lambda)$ $(-\pi \le \lambda \le \pi)$. The matrix $\underset{\sim}{f}(\lambda)$ is Hermitian and is assumed to be positive definite for each λ.

Denote a time series by $\underset{\sim}{X} = (\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_T)$ and let $\underset{\sim}{x} = \mathrm{vec}(\underset{\sim}{X})$ be the $mT \times 1$ vector obtained by arranging the columns of $\underset{\sim}{X}$, from left to right, under one another in a single column. If $\underset{\sim}{\Sigma}$ denotes the $mT \times mT$ covariance matrix of $\underset{\sim}{x}$, we write

$$\underset{\sim}{x}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{x} = \sum_{s,t=1}^{T} \underset{\sim}{x}_s' \underset{\sim}{\Sigma}^{(s,t)}\underset{\sim}{x}_t.$$

The $m \times m$ submatrices $\Sigma^{(s,t)}$ are approximated by

$$\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} e^{i\lambda(s-t)} f^{-1}(\lambda)\,d\lambda \qquad (s,t=1,\ldots,T),$$

and this leads to approximation of $x'\Sigma^{-1}x$ by

$$\frac{T}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr} f^{-1}(\lambda) I(\lambda)\,d\lambda,$$

where

$$I(\lambda) = \frac{1}{2\pi T} \sum_{s,t=1}^{T} e^{i\lambda(s-t)}\, x_s x_t' \qquad (-\pi \le \lambda \le \pi)$$

is the periodogram. The covariance determinant $|\Sigma|$ is approximately the determinant of

$$2\pi \operatorname{diag}\left\{ f\left(\frac{2\pi}{T}\right),\ f\left(\frac{4\pi}{T}\right),\ldots, f(2\pi) \right\} .$$

Thus an approximate Gaussian likelihood for x is

$$(2\pi)^{-mT} \exp\left[-\frac{T}{4\pi} \int_{-\pi}^{\pi} \left\{ \log|f(\lambda)| + \operatorname{tr} f^{-1}(\lambda) I(\lambda) \right\} d\lambda \right] , \qquad (10)$$

in analogy to (3). (See Whittle, 1953b, Theorem 6.) Since $f(-\lambda) = f'(\lambda)$ and $I(-\lambda) = I'(\lambda)$, we may rewrite (10) as

$$(2\pi)^{-mT} \exp\left[-\frac{T}{4\pi} \int_{0}^{\pi} \left\{ 2 \log|f(\lambda)| + \operatorname{tr}\left(f^{-1}(\lambda) I(\lambda) + f'^{-1}(\lambda) I'(\lambda)\right) \right\} d\lambda \right] . \quad (11)$$

Now let

$$\underset{\sim}{f}(\lambda) = \sum_{j=1}^{q} \underset{\sim}{f}_j \, c_j(\lambda) \qquad (0 \leq \lambda \leq \pi), \tag{12}$$

where $c_j(\lambda)$ is defined below (5) and the Hermitian matrices $\underset{\sim}{f}_j$ are positive definite. Denote

$$\underset{\sim}{f}_j = (f_{jk\ell}) = (\bar{f}_{j\ell k}) = \underset{\sim}{f}_j^* = (f^R_{jk\ell} + i\, f^I_{jk\ell}),$$

where the bar designates complex conjugate and the asterisk conjugate transpose. The notation $(\underset{\sim}{A})_{jk}$ designates the element in row j, column k of the matrix $\underset{\sim}{A}$.

Substituting (12) into (11), we obtain

$$(2\pi)^{-mT} \exp\left[- \frac{T}{4\pi} \sum_{j=1}^{q} \left\{ 2 \log |\underset{\sim}{f}_j| \int_0^{\pi} c_j(\lambda)d\lambda \right.\right.$$

$$\left.\left. + \int_0^{\pi} \operatorname{tr}(\underset{\sim}{f}_j^{-1} \underset{\sim}{I}(\lambda) + \underset{\sim}{f}_j'^{-1}\underset{\sim}{I}'(\lambda))\, c_j(\lambda)d\lambda \right\} \right] .$$

If L again denotes the approximate log likelihood and $\underset{\sim}{E}_{jk}$ is an $m \times m$ matrix with a 1 in row j, column k and 0's elsewhere, then

$$\frac{\partial L}{\partial f_{jkk}} = - \frac{T}{2\pi} \operatorname{tr}(\underset{\sim}{f}_j^{-1}\underset{\sim}{E}_{kk}) \int_0^{\pi} c_j(\lambda)d\lambda$$

$$+ \frac{T}{4\pi} \int_0^{\pi} \operatorname{tr}\{\underset{\sim}{f}_j^{-1}\underset{\sim}{E}_{kk}\underset{\sim}{f}_j^{-1} \underset{\sim}{I}(\lambda) + \underset{\sim}{f}_j'^{-1}\underset{\sim}{E}_{kk}\underset{\sim}{f}_j'^{-1}\underset{\sim}{I}'(\lambda)\}\, c_j(\lambda)d\lambda$$

$$= - \frac{T}{2\pi} (\underset{\sim}{f}_j^{-1})_{kk} \int_0^{\pi} c_j(\lambda)d\lambda$$

$$+\frac{T}{4\pi}\int_0^{\pi}\left[\underline{f}_j^{-1}\,\underline{I}(\lambda)\underline{f}_j^{-1}+\{\underline{f}_j^{-1}\,\underline{I}(\lambda)\underline{f}_j^{-1}\}'\right]_{kk} c_j(\lambda)\,d\lambda$$

$$(k=1,\ldots,m,\ j=1,\ldots,q).$$

The other derivatives are

$$\frac{\partial L}{\partial f_{jk\ell}^{R}} = -\frac{T}{\pi}(\underline{f}_j^{-1})_{k\ell}^{R}\int_0^{\pi} c_j(\lambda)\,d\lambda$$

$$+\frac{T}{2\pi}\int_0^{\pi}\left[\underline{f}_j^{-1}\,\underline{I}(\lambda)\underline{f}_j^{-1}+\{\underline{f}_j^{-1}\,\underline{I}(\lambda)\underline{f}_j^{-1}\}'\right]_{k\ell}^{R} c_j(\lambda)\,d\lambda,$$

$$\frac{\partial L}{\partial f_{jk\ell}^{I}} = \frac{T}{\pi}(\underline{f}_j^{-1})_{k\ell}^{I}\int_0^{\pi} c_j(\lambda)\,d\lambda$$

$$-\frac{T}{2\pi}\int_0^{\pi}\left[\underline{f}_j^{-1}\,\underline{I}(\lambda)\underline{f}_j^{-1}-\{\underline{f}_j^{-1}\underline{I}(\lambda)\underline{f}_j^{-1}\}'\right]_{k\ell}^{I} c_j(\lambda)\,d\lambda$$

$$(k\neq\ell,\ k,\ \ell=1,\ldots,m,\ j=1,\ldots,q).$$

Since $\underline{f}_j^{-1}\,\underline{I}(\lambda)\underline{f}_j^{-1}$ is Hermitian, the above derivative expressions may be simplified. When the resulting forms are set equal to 0, the equations become

$$\hat{\underline{f}}_j^{-1}\int_0^{\pi} c_j(\lambda)\,d\lambda = \int_0^{\pi}\hat{\underline{f}}_j^{-1}\underline{I}(\lambda)\hat{\underline{f}}_j^{-1}c_j(\lambda)\,d\lambda$$

$$(j=1,\ldots,q),$$

and the solution is

$$\hat{f}_j = \frac{\int_0^{\pi} I(\lambda) c_j(\lambda) d\lambda}{\int_0^{\pi} c_j(\lambda) d\lambda} = \frac{1}{\lambda_j - \lambda_{j-1}} \int_{\lambda_{j-1}}^{\lambda_j} I(\lambda) d\lambda$$

$$(j=1,\ldots,q).$$

The asymptotic properties noted for the scalar case generalize to the vector case.

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